Large-order asymptotes of the quantum-field expansions for the Kraichnan model of passive scalar advection

2006 J. Phys. A: Math. Gen. 39 7801

(http://iopscience.iop.org/0305-4470/39/25/S02)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **39** (2006) 7801–7813 doi:10.1088/0305-4470/39/25/S02

Large-order asymptotes of the quantum-field expansions for the Kraichnan model of passive scalar advection

A Yu Andreanov, M V Komarova and M Yu Nalimov

Department of Statistical Physics, St Petersburg State University, Uljanovskaja 1, St Petersburg, Petrodvorez, 198504 Russia

Received 22 November 2005, in final form 10 January 2006
Published 7 June 2006
Online at stacks.iop.org/JPhysA/39/7801

Abstract

A simple model of a passive scalar quantity advected by a Gaussian non-solenoidal ('compressible') velocity field is considered. Large-order asymptotes of quantum-field expansions are investigated by the instanton approach. The existence of a finite convergence radius of the series is proved, the type and the position of the singularity of the series in a regularization parameter ε are determined. Anomalous exponents of the main contributions to the structural functions are resummed using new information about the series convergence and two known orders of the ε expansion.

PACS numbers: 47.10.+g, 47.27.Gs, 05.40.+j

1. Introduction

A model of a passive advection of a scalar admixture by a Gaussian short-correlated velocity field, introduced by Obukhov [1] and Kraichnan [2] now attracts considerable interest. Some first structural and response functions in this model demonstrate an anomalous scaling behaviour and the corresponding anomalous exponents can be calculated explicitly as within regular expansions in different small parameters as using numerical simulations. Thus this model provides a good testing ground for various concepts and methods of the turbulence theory like closure approximations, refined similarity relations, Monte-Carlo simulations and renormalization group investigation.

We will discuss the last approach. The renormalization group produces results in a form of some expansions and only a few first terms are known analytically. Different resummation techniques are used to obtain reliable results [3]. The large-order asymptotic information of the perturbation series for field-theoretic models is the base of critical exponents and scaling functions series resummation [3]. The aim of this paper is to develop the instanton approach for large-order asymptotic analysis in Kraichnan dynamic model.

The advection of a passive scalar field $\varphi(\mathbf{x},t)$ is described by a stochastic equation

$$\partial_t\varphi - \nu_0\Delta\varphi + g\boldsymbol{\partial}_i(\mathbf{v}_i\varphi) = \xi(\mathbf{x},t), \tag{1}$$

where $\partial_t = \partial/\partial t$, $\boldsymbol{\partial}_i = \partial/\partial\mathbf{x}_i$, ν_0 is a molecular diffusivity coefficient, Δ is Laplace operator with respect to $\mathbf{x}$, ξ is an artificial Gaussian scalar noise with zero mean and given correlator D_ξ, $\mathbf{v}(\mathbf{x},t)$ is a velocity field, g is a coupling constant; the sum in repeating indices here and henceforth are implied. In Kraichnan model, $\mathbf{v}(\mathbf{x},t)$ obeys a Gaussian distribution with zero average and the following correlator [8]:

$$\tilde{\mathbf{D}}_v^{ij}(\mathbf{x}-\mathbf{x}', t-t') \equiv \langle \mathbf{v}_i(\mathbf{x},t)\mathbf{v}_j(\mathbf{x}',t')\rangle \equiv \mathbf{D}_v^{ij}(\mathbf{x}-\mathbf{x}')\delta(t-t')$$
$$= D_0\delta(t-t')\int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^d}\frac{P_{ij}^{\perp}(\mathbf{k}) + \alpha P_{ij}^{\parallel}(\mathbf{k})}{(\mathbf{k}^2+m^2)^{d/2+\varepsilon}}\,\mathrm{e}^{\mathrm{i}\mathbf{k}(\mathbf{x}-\mathbf{x}')}, \tag{2}$$

where $P^{\perp}(\mathbf{k})$, $P^{\parallel}(\mathbf{k})$ are projectors with respect to vector $\mathbf{k}$ direction, d is a space dimension, ε is the regular expansion parameter and an ultraviolet (UV) regularizator of the model, while m is an infrared regularizator; α is a degree of compressibility: a coefficient representing non-solenoidal modes contribution, D_0 is an arbitrary amplitude regular in ε. The tensor indices of correlator will be omitted and implied henceforth. Expressions (1, 2) describe passive scalar advection due to the compressible fluid turbulent field. Some specific features of the compressible models were studied in [4–6]. The small distance behaviour of the correlation functions for ($d = 1$)-dimensional model in the convective range was investigated in [7]. Article [8] was devoted to the calculation of the relevant composite operators anomalous exponents in the ε and ε^2 orders of ε expansion using the renormalization group approach. An exact anomalous exponent for pair-correlation function was calculated in this paper using the exact solution for the single-time correlator. This result was confirmed in [9, 10] where in particular the anomalous result for tracer model was obtained. The aim of our paper is to investigate large-order properties of the ε series for the anomalous exponents of the composite operators in the convective range in the model (1, 2) obtained by the renormalization group approach and the improvement of these series. The exact anomalous exponent [8] for the pair correlation function will be used to control the results obtained.

It was stated recently [11, 12] the absence of an instanton within the MSR approach [13] in Kraichnan model and it is the Lagrangian variables that should be used at the steepest-descent study of the structural functions $\langle[\varphi(x_1,t_1) - \varphi(x_2,t_2)]^n\rangle$ in the large n limit. We deal with another problem here, namely the large-order asymptotic investigation for the perturbation series of the anomalous dimensions for arbitrary n. Thus we consider the divergent parts at $\varepsilon \to 0$ of the correlation functions investigated in contrast to [11, 12]. Generally different problem leads to the differing instanton solution. Nevertheless, we also introduce the Lagrangian variables following [11] to construct an instanton.

Usually the perturbation series in the quantum-field theories are the asymptotic ones with zero radius of convergence [3]. The situation is quite different in Kraichnan model: the exact result for φ^2 composite operator shows that these series have a finite convergence radius [8]. In this work we establish the following behaviour of the Nth term of the quantum-field series in Kraichnan model in the large N limit:

$$\gamma^{(N)} \sim \frac{N^b}{a^N}, \tag{3}$$

where b characterizes the type of singularity and a fixes the radius of convergence. Our aim is to calculate a and b coefficients for anomalous dimensions of composite operators φ^n. The large N behaviour combined with the known perturbative expansion allows us to extract singularities from the series and to improve the convergence of the quantum-field expansions.

Reference [11] results in an interesting statement about the saturation of the scaling dimensions Δ_n of the φ^n operators. Namely, they are predicted to have the finite limit value at $n \to \infty$. One of our purposes is to verify this statement for the compressible case on the basis of the results of the resummation for two orders of ε expansion. The saturation effect has been observed in Navier–Stokes flows in [16]. So this problem is relevant not only for the Kraichnan model, but also for real flows.

We have to stress our results that are valid for the compressible fluid case only. As concerns the model with the incompressible fluid considered in [11], our instanton solution cannot yield the large-order asymptotes of the ε expansion of the anomalous exponents of the non-trivial composite operators because of the vanishing of the fore-exponential factor at the instanton found.

The paper is organized as follows. In section 2, the method of renormalization constants calculation is discussed. The general stationarity equations in field variables are obtained in section 3. The solution of these equations is described in section 4. In section 5, we deal with the stationarity equations in coupling constant and coordinate arguments. The integration over the scale parameter is described in section 6. Section 7 is devoted to a re-expansion of the anomalous dimensions of the composite operators φ^n. The summary is written in the section 8.

2. Large-order investigation of renormalization constants

The quantity of interest for Kraichnan model is, in particular, the infrared behaviour of the single-time structural functions

$$\langle[\varphi(t,\mathbf{x}) - \varphi(t,\mathbf{x}')]^n\rangle, \qquad r \equiv |\mathbf{x} - \mathbf{x}'|.$$

It is determined by anomalous dimensions γ_{φ^n} of the composite operators φ^n, the former have been calculated up to two orders of ε expansion [8].

Let us consider the response functions $\int \mathrm{d}\mathbf{x}_0\,\mathrm{d}t_0 \langle\varphi^n(\mathbf{x}_0,t_0)\varphi'(\mathbf{x}_1,t_1)\ldots\varphi'(\mathbf{x}_n,t_n)\rangle$ represented in a form

$$\int \mathrm{d}\mathbf{x}_0\,\mathrm{d}t_0 \int D\mathbf{v} G(\mathbf{x}_0,t_0,\mathbf{x}_1,t_1)\ldots G(\mathbf{x}_0,t_0,\mathbf{x}_n,t_n). \tag{4}$$

Large-order investigation of this expression as a series in g is very difficult due to the presence in equation (4) of the additional parameters such as $m, \mathbf{x}_i, t_i, \varepsilon$ that can constitute different combinations comparable with the large parameter N of the steepest-descent method. The same problem for the classical φ^4 static model was described in [14]. Fortunately, as we are interested in the anomalous dimensions γ_{φ^n}, we can limit ourself by the consideration of renormalization constants Z_{φ^n} of the composite operators φ^n

$$\langle\varphi^n\varphi'\ldots\varphi'\rangle^R = Z_{\varphi^n}^{-1}\langle\varphi^n\varphi'\ldots\varphi'\rangle. \tag{5}$$

Constants Z_{φ^n} are m and ν independent in the MS scheme and contain only poles with respect to ε variable. They are connected with the UV divergences of the equal time ($t_1 = \cdots = t_n$) diagrams at zero external momenta. In the framework of ε regularization, that is similar to the dimensional regularization, we can explore an analytical continuation to the $\varepsilon < 0$ region. In this region, it is possible to set $m = 0$ in (2). This yields

$$\mathbf{D_v}^{ij}(\mathbf{x}) = D_0 \frac{2^{-\varepsilon}\Gamma(-\varepsilon/2)}{(4\pi)^{d/2}\Gamma(d/2+\varepsilon/2)(d+\varepsilon)} |\mathbf{x}|^{\varepsilon}\left[(d-1+\alpha+\varepsilon)\delta_{ij} + \varepsilon(\alpha-1)\frac{\mathbf{x}_i\mathbf{x}_j}{\mathbf{x}^2}\right], \quad \varepsilon < 0. \tag{6}$$

The D_0 parameter is fixed up to the finite renormalization that does not affect the critical exponents. Let us then choose it equal to

$$D_0 = \frac{\Gamma(d/2+\varepsilon/2)(d+\varepsilon)2^{\varepsilon}}{\Gamma(1-\varepsilon/2)\Gamma(d/2)d}, \tag{7}$$

so that the first fraction in (6) contains a poles in ε only. It is then convenient to divide the correlator discussed into three parts with different ε dependence

$$\begin{aligned}
&\mathbf{D_v}(\mathbf{x}) = \mathbf{D}(\mathbf{x}) + \boldsymbol{\Delta}(\mathbf{x}) + \bar{\boldsymbol{\Delta}}(\mathbf{x}),\\
&\mathbf{D}^{ij}(\mathbf{x}) = -\frac{2|\mathbf{x}|^{\varepsilon}}{d(4\pi)^{d/2}\Gamma(d/2)}\left(\delta_{ij} + (\alpha-1)\frac{\mathbf{x}_i\mathbf{x}_j}{\mathbf{x}^2}\right),\\
&\boldsymbol{\Delta}^{ij}(\mathbf{x}) = \frac{2(d-1+\alpha)}{d(4\pi)^{d/2}\Gamma(d/2)}\frac{1-|\mathbf{x}|^{\varepsilon}}{\varepsilon}\delta_{ij}, && (8)\\
&\bar{\boldsymbol{\Delta}}^{ij} = -\frac{2(d-1+\alpha)}{d(4\pi)^{d/2}\Gamma(d/2)}\frac{1}{\varepsilon}\delta_{ij}. && (9)
\end{aligned}$$

Reference [14] was devoted to the modification of the steepest-descent approach for the large-order asymptotes investigation of renormalization constants at the example of the well-known static φ^4 model in MS scheme and dimensional regularization. According to this paper UV divergences have a meaning as the perturbation objects and must be treated as the fore-exponential factors. So the factor $\exp(S_{\text{div}})$, where S_{div} absorbs all the divergent terms of the action that has to be expanded in the McLoran series in ε before the application of the steepest-descent approach. As expressions (8) contain the UV divergences (at $\varepsilon \to 0$, $\mathbf{x} \to 0$), one need to include the corresponding terms to the divergent part of the action S_{div} and consider them as fore-exponential factors. In section 6, we will see that these terms affect an amplitude of the asymptotic only.

For φ^2 operator, we can write in the Lagrangian variables using (4), [11]

$$\begin{aligned}
\langle\varphi^2(\mathbf{x}_0,t_0)\varphi'(\mathbf{x}_1,t)\varphi'(\mathbf{x}_2,t)\rangle &= M^2\int D\mathbf{v}\int_{\mathbf{c}_1(t)=\mathbf{x}_1,\mathbf{c}_2(t)=\mathbf{x}_2}^{\mathbf{c}_1(t_0)=\mathbf{c}_2(t_0)=\mathbf{x}_0} D\mathbf{c}_1 D\mathbf{c}_2 D\mathbf{c}_1' D\mathbf{c}_2'\\
&\times\exp\left(-\frac{1}{2}\mathbf{v}\tilde{D}_v^{-1}\mathbf{v} - \nu Z_\nu \mathbf{c}_1'^2 - \nu Z_\nu \mathbf{c}_2'^2 + \mathrm{i}\mathbf{c}_1'\dot{\mathbf{c}}_1 + \mathrm{i}\mathbf{c}_2'\dot{\mathbf{c}}_2 + \mathrm{i}g\mathbf{c}_1'\mathbf{v}(\mathbf{c}_1) + \mathrm{i}g\mathbf{c}_2'\mathbf{v}(\mathbf{c}_2)\right)\\
&\times\left[\int D\mathbf{v}\exp\left(-\frac{1}{2}\mathbf{v}\tilde{D}_v^{-1}\mathbf{v}\right)\right]^{-1},
\end{aligned} \tag{10}$$

$$M \equiv \frac{1}{(4\pi\nu T)^{d/2}}\left(\int_{\mathbf{c}(t)=0}^{\mathbf{c}(t_0)=0} D\mathbf{c}D\mathbf{c}'\exp(-\nu\mathbf{c}'^2 + \mathrm{i}\mathbf{c}'\dot{\mathbf{c}})\right)^{-1}, \tag{11}$$

where $T \equiv t_0 - t$. The integrations in times of the fields and the sum over the vectorial indices of the fields and the tensor $\mathbf{D}$ are implied here and henceforth. Integration of (10) in $\mathbf{v}$ field yields

$$\begin{aligned}
M^2\int_{\mathbf{c}_1(t)=\mathbf{x}_1,\mathbf{c}_2(t)=\mathbf{x}_2}^{\mathbf{c}_1(t_0)=\mathbf{c}_2(t_0)=\mathbf{x}_0} & D\mathbf{c}_1 D\mathbf{c}_2 D\mathbf{c}_1' D\mathbf{c}_2' \exp\big(-\nu Z_\nu\big(\mathbf{c}_1'^2 + \mathbf{c}_2'^2\big)\\
&+\mathrm{i}\mathbf{c}_1'\dot{\mathbf{c}}_1 + \mathrm{i}\mathbf{c}_2'\dot{\mathbf{c}}_2 - u\mathbf{c}_1'[\mathbf{D}(\mathbf{c}_1-\mathbf{c}_2) + \boldsymbol{\Delta}(\mathbf{c}_1-\mathbf{c}_2) + \bar{\boldsymbol{\Delta}}]\mathbf{c}_2'\big).
\end{aligned} \tag{12}$$

Due to the $\delta(t-t')$-type time dependence of $\tilde{\mathbf{D}}_\mathbf{v}$ (2) fields $\mathbf{c}_1$ and $\mathbf{c}_2$ in $\mathbf{D}(\mathbf{c}_1-\mathbf{c}_2)$, $\boldsymbol{\Delta}(\mathbf{c}_1-\mathbf{c}_2)$ have an coinciding time arguments. It is worth while to scale u by ν to get a new fully dimensionless coupling constant u. After the expansion of the factor S_{div} containing the UV divergent part of action

$$S_{\text{div}} = -u\nu\mathbf{c}_1'\boldsymbol{\Delta}(\mathbf{c}_1-\mathbf{c}_2)\mathbf{c}_2' - u\nu\mathbf{c}_1'\bar{\boldsymbol{\Delta}}(\mathbf{c}_1-\mathbf{c}_2)\mathbf{c}_2' - \nu(Z_\nu-1)\big(\mathbf{c}_1'^2 + \mathbf{c}_2'^2\big)$$

(12) takes the form

$$M^2 \int_{\mathbf{c}_1(t)=\mathbf{x}_1,\mathbf{c}_2(t)=\mathbf{x}_2}^{\mathbf{c}_1(t_0)=\mathbf{c}_2(t_0)=\mathbf{x}_0} D\mathbf{c}_1 D\mathbf{c}_2 D\mathbf{c}'_1 D\mathbf{c}'_2 A^{[2]} B^{[2]}\, \mathrm{e}^{-S^{[2]}}, \tag{13}$$

where

$$S^{[2]} = \nu\left(\mathbf{c}_1'^2 + \mathbf{c}_2'^2\right) - \mathrm{i}\mathbf{c}'_1\dot{\mathbf{c}}_1 - \mathrm{i}\mathbf{c}'_2\dot{\mathbf{c}}_2 + u\nu\mathbf{c}'_1\mathbf{D}(\mathbf{c}_1 - \mathbf{c}_2)\mathbf{c}'_2, \tag{14}$$

$$A^{[2]} = \sum_{j=0}^{\infty} \frac{(-u\nu\mathbf{c}'_1\boldsymbol{\Delta}(\mathbf{c}_1 - \mathbf{c}_2)\mathbf{c}'_2)^j}{j!}, \tag{15}$$

$$B^{[2]} = \sum_{j=0}^{\infty} \frac{\left(-u\nu\mathbf{c}'_1\bar{\boldsymbol{\Delta}}\mathbf{c}'_2 - \nu(Z_\nu - 1)\left(\mathbf{c}_1'^2 + \mathbf{c}_2'^2\right)\right)^j}{j!}. \tag{16}$$

These expressions can be generalized for $\langle\varphi^n(\mathbf{x}_0, t_0)\varphi'(\mathbf{x}_1, t)\ldots\varphi'(\mathbf{x}_n, t)\rangle$ Green function. It is represented by the path integral in $\mathbf{c}_i$, $\mathbf{c}'_i$ fields ($i = 1, \ldots, n$) with the action

$$S^{[n]} = \nu\sum_{i=1}^{n}\mathbf{c}_i'^2 - \mathrm{i}\sum_{i=1}^{n}\mathbf{c}'_i\dot{\mathbf{c}}_i + \frac{u\nu}{2}\sum_{i\neq l}\mathbf{c}'_i\mathbf{D}(\mathbf{c}_i - \mathbf{c}_l)\mathbf{c}'_l. \tag{17}$$

The normalization factor equals to M^n now. The boundary conditions are $\mathbf{c}_i(t_0) = \mathbf{x}_0$, $\mathbf{c}_i(t) = \mathbf{x}_i$. The pre-exponential factors $A^{[n]}$ and $B^{[n]}$ are similar to (15, 16) up to the additional sums over i index of the fields $\mathbf{c}_i$, $\mathbf{c}'_i$. Note that the latter play a role of generalized coordinates and momenta for the action $S^{[n]}$. So we can represent our system with a set of n moving quasi-particles.

Now let us discuss the connection between the Green functions considered and the critical exponents. The singularities of the Green function and of its renormalization constant are related by the identity (5). The renormalization constant Z_{φ^n} determines uniquely the anomalous dimension γ_{φ^n}.

The left-hand side of (5) is free of singularities and, consequently, so does the right one: the singularities of Z_{φ^n} are cancelled by those of non-renormalized function $\langle\varphi^n\varphi'\ldots\varphi'\rangle$. We could use this to compute the singularities of $Z^{(N)}_{\varphi^n}$ by stating that

$$Z^{(N)}_{\varphi^n}\langle\varphi^n\varphi'\ldots\varphi'\rangle^{(0)} + Z^{(N-1)}_{\varphi^n}\langle\varphi^n\varphi'\ldots\varphi'\rangle^{(1)} + \cdots + Z^{(0)}_{\varphi^n}\langle\varphi^n\varphi'\ldots\varphi'\rangle^{(N)} \tag{18}$$

is finite. Henceforth, $X^{(N)}$ denotes the Nth order of the expansion for the value X in u. Usually expansions in the quantum-field theories are the asymptotic ones with an exponential growth of the coefficients. Such growth would make all terms in (18) except the first and the last ones irrelevant as $N \to \infty$. Then equation (18) could be easily resolved for $Z^{(N)}$ coefficients. The existence of the non-zero radius of convergence will be shown for Kraichnan model. Then a large set of terms in (18) becomes relevant and we face the problem of calculating of all the terms $\langle\varphi^n\varphi'\ldots\varphi'\rangle^{(i)}$, $i < N$: we cannot pick out all the singularities present in (18).

That is why we shall use the identity

$$\ln\langle\varphi^n\varphi'\ldots\varphi'\rangle_R = -\ln Z_{\varphi^n} + \ln\langle\varphi^n\varphi'\ldots\varphi'\rangle.$$

The lhs of the expression is finite again at $\varepsilon \to 0$. To restore the Nth coefficient of the expansion for $\ln Z_{\varphi^n}$ in u, we will investigate the residue in ε for McLoran expansion of $\ln\langle\varphi^n\varphi'\ldots\varphi'\rangle$. The 'replica trick' that is based on the identity

$$\ln\langle\varphi^n\varphi'\ldots\varphi'\rangle = \lim_{L\to 0}\frac{\partial}{\partial L}\langle\varphi^n\varphi'\ldots\varphi'\rangle^L,$$

will be used to treat $\ln\langle\varphi^n\varphi'\ldots\varphi'\rangle$. The Green function in L-power in the last expression is then substituted by the path integral representation of $\langle\varphi^n\varphi'\ldots\varphi'\rangle$ with all integration variables considered now as L-dimensional ones.

3. Instanton analysis and immovable particles

As in classical works [3, 15], we add an integration $\oint \mathrm{d}u/u^{N+1}$ to produce the Nth order of the perturbation expansion. Summarizing all the remarks mentioned above the Nth order term of the expansion of $\ln Z_{\varphi^n}$ in u can be written as

$$(\ln\langle\varphi^n\varphi'\ldots\varphi'\rangle)^{(N)} = \frac{1}{2\pi i}\lim_{L\to 0}\frac{\partial}{\partial L}\operatorname*{residue}_{\varepsilon=0}\oint\frac{\mathrm{d}u}{u^{N+1}}\prod_{\beta=1}^{L}\int \mathrm{d}T_\beta\int \mathrm{d}\mathbf{x}_{0\beta}$$
$$\times\int d(\mathbf{x}_{2\beta}-\mathbf{x}_{1\beta})\int d(\mathbf{x}_{3\beta}-\mathbf{x}_{1\beta})\ldots\int d(\mathbf{x}_{n\beta}-\mathbf{x}_{1\beta})M_\beta^n$$
$$\times\int_{\{\mathbf{c}_{i\beta}(t_\beta)=\mathbf{x}_{i\beta}\}}^{\{\mathbf{c}_{i\beta}(t_{0\beta})=\mathbf{x}_{0\beta}\}}\left(\prod_{i=1}^{n}D\mathbf{c}_{i\beta}D\mathbf{c}'_{i\beta}\right)\mathbf{A}_\beta^{[n]}\mathbf{B}_\beta^{[n]}\exp\left(-S_\beta^{[n]}-\mathrm{i}\mathbf{k}(\mathbf{x}_{2\beta}-\mathbf{x}_{1\beta})\right) \qquad (19)$$

with the action $S_\beta^{[n]}$ given by (17). The variables $T, \mathbf{x}_i, \mathbf{c}_i, \mathbf{c}'_i$ have now an additional replica index $\beta = 1, \ldots, L$.

Let us shift the variables to eliminate all the dependences of the path integral limits in coordinates

$$\mathbf{c}_{i\beta}(\tau_\beta) = \bar{\mathbf{c}}_{i\beta}(\tau_\beta) + \mathbf{x}_{i\beta} - \frac{(\mathbf{x}_{i\beta}-\mathbf{x}_{0\beta})(\tau_\beta - t_\beta)}{T_\beta}, \qquad (20)$$

the new fields $\bar{\mathbf{c}}_{i\beta}$ have zero boundary conditions. Sometimes we will return to the original notation $\mathbf{c}_i$ and we will omit the index β for brevity.

The steepest-descent approach must be applied to expression (19) with respect to all integrations except the one over the scale parameter of the model that has an essential non-instanton form. Indeed, overall UV divergences of interest arise from the integration over the scale parameter and can be extracted with the help of the integration by parts. We will show that at the instanton solution the value $y = |\mathbf{x}_{2\beta} - \mathbf{x}_{1\beta}|$ does not depend actually on β and turn out to be the scale parameter.

We proceed now to the finding of the saddle-point of the action in (19). Let us scale the variables to figure out the N dependence of the action. The renormalization constants considered are independent on renormalized diffusivity ν and $\mathbf{k}$ momentum so they can be also scaled

$$\mathbf{c}'_i \to N\mathbf{c}'_i \quad \mathbf{k} \to N\mathbf{k}, \qquad \nu = \eta/N \qquad (21)$$

(we perform this scaling both for (19) and for the factor M integral representation (11); the Jacobians arriving are constant and cancel out mutually). Such a scaling produces $N^{ndL/2}$ factor for the expression considered. The value $S_\beta^{[n]}$ has now the following form:

$$S_\beta^{[n]} = N\left\{\eta\sum_{i=1}^{n}\mathbf{c}_i'^2 - \mathrm{i}\sum_{i=1}^{n}\mathbf{c}'_i\dot{\mathbf{c}}_i + \frac{u\eta}{2}\sum_{i\neq j}\mathbf{c}'_i D(\mathbf{c}_i-\mathbf{c}_j)\mathbf{c}'_j\right\} \equiv N\widetilde{S}_\beta^{[n]},$$

all the fields are assumed to have the replica index β. In the framework of the instanton approach, we vary the functional

$$S \equiv N\sum_\beta\left(\widetilde{S}_\beta^{[n]} + \mathrm{i}\mathbf{k}(\mathbf{x}_{2\beta}-\mathbf{x}_{1\beta})\right) + N\ln u$$

with respect to all the variables. The variations in $\bar{\mathbf{c}}_m, \mathbf{c}'_m$ yield

$$\frac{\delta S}{\delta\bar{\mathbf{c}}_m} = 0 \quad\Rightarrow\quad -\mathrm{i}\dot{\mathbf{c}}'_m = u\eta\sum_{\substack{l\\ l\neq m}}\mathbf{c}'_m\frac{\partial D(\mathbf{c}_m-\mathbf{c}_l)}{\partial(\mathbf{c}_m-\mathbf{c}_l)}\mathbf{c}'_l, \qquad (22)$$

$$\frac{\delta S}{\delta \mathbf{c}'_m} = 0 \quad \Rightarrow \quad \mathrm{i}\dot{\mathbf{c}}_m = u\eta \sum_{\substack{l \\ l\neq m}} D(\mathbf{c}_m - \mathbf{c}_l)\mathbf{c}'_l + 2\eta \mathbf{c}'_m. \tag{23}$$

The dependence on coordinates $\mathbf{x}_i$ is represented in the functional S only by the fields $\mathbf{c}_i, \mathbf{c}_j$ (see (20)). So the variation in $\mathbf{x}_0$ yields the full momentum conservation law for the quasi-particles

$$\frac{\delta S}{\delta \mathbf{x}_0} = 0 \quad \Rightarrow \quad \sum_{l=1}^{n} \mathbf{c}'_l = 0.$$

The variations with respect to the coordinates $(\mathbf{x}_m - \mathbf{x}_1)$ $(m \geqslant 3)$ yield

$$\frac{\mathrm{i}}{T}\int_t^{t_0} \mathrm{d}\tau\, \mathbf{c}'_m + u\eta \int_t^{t_0} \mathrm{d}\tau\, \frac{t_0-\tau}{T} \sum_{\substack{l \\ l\neq m}} \mathbf{c}'_m \frac{\partial D(\mathbf{c}_m - \mathbf{c}_l)}{\partial(\mathbf{c}_m - \mathbf{c}_l)} \mathbf{c}'_l = 0, \qquad m \geqslant 3. \tag{24}$$

Combining together (22) and (24), we get the equation

$$\frac{1}{T}\int_t^{t_0} \mathrm{d}\tau\, \mathbf{c}'_m - \frac{1}{T}\int_t^{t_0} \mathrm{d}\tau (t_0 - \tau)\dot{\mathbf{c}}'_m = \mathbf{c}'_m(t) = 0. \tag{25}$$

The last expression can be considered as a boundary condition at $\tau = t$ on the field $\mathbf{c}'_m(\tau)$ $(m \geqslant 3)$. (22) is a first-order differential equation with respect to $\mathbf{c}'_m$ and has a zero boundary condition (25). It has then the trivial solution

$$\mathbf{c}'_m(\tau) = 0, \qquad t \leqslant \tau \leqslant t_0 \qquad m \geqslant 3, \quad \beta = 1, \ldots, L \tag{26}$$

that is locally unique. Equation (23) provides then

$$\dot{\mathbf{c}}_m = 0 \qquad \mathbf{c}_m(\tau) = \mathbf{x}_m = \mathbf{x}_0 \qquad m \geqslant 3, \quad \beta = 1, \ldots, L. \tag{27}$$

This means that the instanton sought for implies only a couple of quasi-particles in motion while the other stay at the point $\mathbf{x}_0$. The saddle-point equations reduce then to the $n = 2$ case with only four non-trivial fields $\bar{\mathbf{c}}_1, \bar{\mathbf{c}}_2, \mathbf{c}'_1, \mathbf{c}'_2$ (replica index β is assumed). Such a skewness of the instanton is explained by the term $\mathbf{k}(\mathbf{x}_{2\beta} - \mathbf{x}_{1\beta})$ of the variable action. Indeed, as we are interested in the renormalization constants that do not depend on a conjugate momentum running through the diagrams of the response function considered we have set all momenta except $\mathbf{k}$ in (19) equal to zero. The momentum $\mathbf{k}$ cannot be zero since the response function would diverge then. Note that this divergence is unrelated to the problem as it has a trivial power form and does not affect on the radius of convergence analysed. Nevertheless, the accurate treatment requires $\mathbf{k} \neq 0$ that causes the skewness obtained.

Thus the analysis of the stationarity equations demonstrates the trivial solution (26, 27) for the variables $\mathbf{x}_k$, $k \geqslant 3$. The problem reduces then to the φ^2 case and we can limit ourselves to the study of the saddle-point of the following non-trivial integral representation

$$\frac{1}{2\pi \mathrm{i}} \oint \frac{\mathrm{d}u}{u^{N+1}} \prod_\alpha \int \mathrm{d}\mathbf{x}_0 \int \mathrm{d}\mathbf{x} \int_0^\infty \mathrm{d}T \int D\bar{\mathbf{c}}_1 D\bar{\mathbf{c}}_2 D\mathbf{c}'_1 D\mathbf{c}'_2 A^{[2]} B^{[2]} M^2\, \mathrm{e}^{-S_\alpha^{[2]}}, \tag{28}$$

that represents the case $n = 2$. Indeed, the substitution of (22, 23) to the expressions for $A_\beta^{[n]}, B_\beta^{[n]}$ reduces their values to the two-particle case $A^{[2]}, B^{[2]}$ and $S_\beta^{[n]}$ as well.

Let us introduce new variables: $\mathbf{p} = \mathbf{c}'_1 - \mathbf{c}'_2, \mathbf{P} = \mathbf{c}'_1 + \mathbf{c}'_2, \mathbf{q} = \mathbf{c}_1 - \mathbf{c}_2, \mathbf{Q} = \mathbf{c}_1 + \mathbf{c}_2, \bar{\mathbf{q}} = \bar{\mathbf{c}}_1 - \bar{\mathbf{c}}_2, \bar{\mathbf{Q}} = \bar{\mathbf{c}}_1 + \bar{\mathbf{c}}_2$. The fields $\bar{\mathbf{q}}, \bar{\mathbf{Q}}$ satisfy zero boundary conditions. The variables introduced correspond to the centre-mass frame of reference of the quasi-particles. Expression (28) transforms into

$$\frac{1}{2\pi \mathrm{i}} \oint \frac{\mathrm{d}u}{u^{N+1}} \int \mathrm{d}\mathbf{x}_0 \int \mathrm{d}\mathbf{x} \int_0^\infty \mathrm{d}T\, M^2 A^{[2]} B^{[2]} \int D\mathbf{p} D\bar{\mathbf{q}} D\mathbf{P} D\bar{\mathbf{Q}}\, \mathrm{e}^{-N\widetilde{S}}, \tag{29}$$

$$\widetilde{S} = \sum_{\beta=1}^{L} \left(\widetilde{S}_{\beta}^{[2]} + \mathrm{i}\mathbf{k}\mathbf{x}_{\beta}\right) + \ln u, \qquad \mathbf{x} = \mathbf{x}_2 - \mathbf{x}_1, \tag{30}$$

$$\widetilde{S}_{\beta}^{[2]} = -\frac{\mathrm{i}}{2}\mathbf{p}\left(\dot{\bar{\mathbf{q}}} + \frac{\mathbf{x}}{T}\right) - \frac{\mathrm{i}}{2}\mathbf{P}\left(\dot{\bar{\mathbf{Q}}} + \frac{2\mathbf{x}_0 - \mathbf{x}_1 - \mathbf{x}_2}{T}\right) + \frac{u\eta}{4}(\mathbf{P} + \mathbf{p})D(\mathbf{q})(\mathbf{P} - \mathbf{p}). \tag{31}$$

All the variables in the right-hand side of (31) are assumed to have the replica index β. Zero conditions on variations of the action $\widetilde{S}$ in $\bar{\mathbf{q}}$, $\bar{\mathbf{Q}}$, $\mathbf{p}$, $\mathbf{P}$, $\mathbf{x}$, $\mathbf{x}_0$ and u give the instanton equations.

It is convenient to carry out the calculation in two steps. First we solve the instanton equations in fields $\mathbf{P}$, $\bar{\mathbf{Q}}$, $\mathbf{p}$, $\bar{\mathbf{q}}$ and the variable $\mathbf{x}_0$. We use this solution to simplify the functional $\widetilde{S}$. Then we vary the simplified functional $\widetilde{S}$ in $\mathbf{x}_\beta$ and u we obtain saddle-point equations in a simplified manner and we turn out to be successful in solving them.

4. Two quasi-particle saddle-point solution

The saddle-point of the action $\widetilde{S}$ with respect to $\mathbf{x}_0$, $\bar{\mathbf{Q}}$, $\mathbf{P}$, $\bar{\mathbf{q}}$, $\mathbf{p}$ is determined by the following equations:

$$\frac{\delta\widetilde{S}}{\delta\mathbf{x}_0} = 0 \quad\Rightarrow\quad \dot{\mathbf{P}} = 0,$$

$$\frac{\delta\widetilde{S}}{\delta\bar{\mathbf{Q}}} = 0 \quad\Rightarrow\quad \mathbf{P} = 0,$$

$$\frac{\delta\widetilde{S}}{\delta\mathbf{P}} = 0 \quad\Rightarrow\quad \mathrm{i}\dot{\mathbf{Q}} = u\eta\mathbf{D}(\mathbf{q})\mathbf{P} + 2\eta\mathbf{P},$$

$$\frac{\delta\widetilde{S}}{\delta\bar{\mathbf{q}}} = 0 \quad\Rightarrow\quad \mathrm{i}\dot{\mathbf{p}} = \frac{u\eta}{2}\frac{\partial}{\partial\mathbf{q}}[\mathbf{p}\mathbf{D}(\mathbf{q})\mathbf{p} - \mathbf{P}\mathbf{D}(\mathbf{q})\mathbf{P}],$$

$$\frac{\delta\widetilde{S}}{\delta\mathbf{p}} = 0 \quad\Rightarrow\quad \mathrm{i}\dot{\mathbf{q}} = 2\eta\mathbf{p} - u\eta\mathbf{D}(\mathbf{q})\mathbf{p}$$

with the following boundary conditions:

$$\mathbf{q}(t) = -\mathbf{x}, \qquad \mathbf{Q}(t) = \mathbf{x}_1 + \mathbf{x}_2,$$
$$\mathbf{q}(t_0) = 0, \qquad \mathbf{Q}(t_0) = 2\mathbf{x}_0.$$

To find the solution, we suppose that the vector $\mathbf{p}$ is parallel to $\mathbf{q}$. In fact, one can show that this is the only solution of the original system satisfying the given boundary conditions. Similar to [11] the simplified system is trivially integrated

$$\mathbf{Q}(\tau) = 2\mathbf{x}_0 = \mathbf{x}_1 + \mathbf{x}_2, \qquad \mathbf{P}(\tau) = 0, \tag{32}$$

$$\mathbf{p}(\tau) = \frac{\mathrm{i}\dot{\mathbf{q}}(\tau)}{2\eta - u\eta D(\mathbf{q}(\tau))}, \qquad \dot{\mathbf{q}}(\tau) = \frac{I_1(\mathbf{x})}{T}\sqrt{2\eta - u\eta D(\mathbf{q}(\tau))}, \tag{33}$$

$$I_1(\mathbf{x}) \equiv \int_0^x \frac{\mathrm{d}z}{\sqrt{2\eta - u\eta D(z)}}, \qquad D(z) \equiv -\frac{2\alpha|z|^{\varepsilon}}{d(4\pi)^{d/2}\Gamma(d/2)}, \tag{34}$$

where $D(z)$ and x are the projections of the tensor $\mathbf{D}(\mathbf{q})$ and of $\mathbf{x}$ to the direction of $\mathbf{p}$ (index β is implied for the time variables T, τ, all fields and the variables x, $\mathbf{x}$).

The functional $\widetilde{S}$ at the saddle-point reads

$$\widetilde{S} = \sum_{\beta=1}^{L} \left(\widetilde{S}_{\beta}^{[2]} - \mathrm{i}\mathbf{k}\mathbf{x}_{\beta}\right) + \ln u, \qquad \widetilde{S}_{\beta}^{[2]} = \frac{I_1^2(\mathbf{x}_\beta)}{4T_\beta}, \tag{35}$$

with the replica index β indicated explicitly.

Let us note that due to $P = 0$ the identity $\mathbf{c}'_1 = -\mathbf{c}'_2$ holds and the $\bar{\Delta}$-term in $B^{[2]}$ factor (13) is cancelled out by the renormalization $(Z_\nu - 1)$ of the diffusivity ν and thus $B^{[2]} \equiv 1$ (16, 8). The pre-exponential factor $A^{[2]}$ at the saddle-point is then given by

$$A_\beta^{[2]} = \sum_{j=0}^{\infty} \frac{1}{j!} \left(-\frac{u\eta N I_1(\mathbf{x}_\beta) I_\Delta(\mathbf{x}_\beta)}{4T_\beta} \right)^j \qquad I_\Delta(\mathbf{x}) = \int_0^x \frac{\Delta(z)\,\mathrm{d}z}{\sqrt{2\eta - u\eta D(z)}^3}. \tag{36}$$

5. A saddle-point solution over u and $\mathbf{x}_\beta$

It is obvious that the variations in u and $\mathbf{x}_\beta$ of $\widetilde{S}$ given by (30) and $\widetilde{S}$ given by (35) are explicitly the same. The form (35) is much more convenient: the variation with respect to u, $\mathbf{x}_\beta$ yields

$$\frac{\delta \widetilde{S}}{\delta u} = 0 \quad \Rightarrow \quad \sum_{\beta=1}^{L} \frac{u\eta I_1(\mathbf{x}_\beta) I_2(\mathbf{x}_\beta)}{4T_\beta} = -1, \tag{37}$$

$$I_2(\mathbf{x}_\beta) = \int_0^{x_\beta} \frac{\mathrm{d}z D(z)}{[\sqrt{2\eta - uD(z)}]^3},$$

$$\frac{\delta \widetilde{S}}{\delta \mathbf{x}_\beta} = 0 \quad \Rightarrow \quad \frac{I_1(\mathbf{x}_\beta)}{2T_\beta \sqrt{2\eta - uD(\mathbf{x}_\beta)}} = \mathrm{i}\mathbf{k}. \tag{38}$$

These equations include T_β parameter: we suppose that the solution exists. Substituting it in expression (35), we derive the action $\widetilde{S}_\beta^{[2]} = N I_1^2 / T_\beta$, i.e. the integration over T_β has the form

$$\int_0^\infty \mathrm{d}T_\beta\, f(T_\beta) \exp\left(-\frac{N I_1^2}{T_\beta} \right).$$

The convergence of this integral is assured by $f(T)$ factor that absorbs all the fluctuation integrals over the fields and $u, \mathbf{x}_0, \mathbf{x}$. It is easy to see that the main contribution to the integration over T_β at $N \to \infty$ comes from the large T_β region.

Let us return to equations (37), (38) analysis. The left-hand side of (38) is non-zero in the $T_\beta \to \infty$ limit only if the value $\sqrt{2\eta - uD(\mathbf{x}_\beta)}$ is small (of order T_β^{-1}). On the other hand, equation (37) is satisfied only for large values of I_2, i.e as the denominator $\sqrt{2\eta - uD(z)}$ tends to zero at the point that tends to $x + 0$ for large T_β. Therefore, the solution of the system (37), (38) can be written at large T_β as

$$x_\beta = y + \frac{\delta_\beta}{T_\beta^2}, \qquad u = \frac{2}{D\big(y + \delta_0 \sum_\beta T_\beta^{-2}\big)} \tag{39}$$

with δ_0, δ_β being some constants and y being an arbitrary scale parameter. The substitution of this solution in equations (37), (38) allows us to calculate δ_0, δ_β constants. In the limit $T \to \infty$ this saddle-point reads in terms of $\mathbf{x}_\beta, u$

$$\mathbf{x}_\beta = y\frac{\mathbf{k}}{|\mathbf{k}|}, \quad \beta = 1, \ldots, L, \qquad u = \frac{2}{D(y)}.$$

The presence of an arbitrary scale parameter y is the usual fact in the instanton analysis of the scale invariant models [15]. The correct work around supposes an insertion of a unit decomposition in the initial expression (19)

$$1 \equiv \int_0^\infty \mathrm{d}y\, \delta(y - |\mathbf{x}_{\beta=1}|).$$

The integration over the scale parameter is then carried out explicitly by the Faddeev–Popov method. It has the non-saddle-point form and its singularities at $\varepsilon \to 0$ determine the renormalization constants and the critical exponents (the same problem arrives for φ^4 model [14]). The δ-function mentioned above plays an important role: it resolves the so-called zero-mode problem [3] of carrying out of the integration over the fluctuations. This produces an additional $\sqrt{N}$ factor as contribution of the 'zero mode' in the numerator of the fluctuation integral. The additional factor $N^{-(ndL+1)/2}$ arrives from the fluctuation around saddle-point integral over the variables $u, \mathbf{x}_0, \mathbf{x}_{i\beta}$ $(i = 2, \ldots, n, \beta = 1, \ldots, L)$. Together with $N^{ndL/2}$ factor arrived from the scaling (21), the total power of N reduces to unit.

The last question we have to discuss in this section is the saddle-point method applicability. It should be mentioned that the fluctuation integration can be carried out at $\varepsilon = 0$. Though the longitudinal projector

$$P^{\parallel}_{ab} = \frac{(\mathbf{c}_i - \mathbf{c}_j)_a(\mathbf{c}_i - \mathbf{c}_j)_b}{(\mathbf{c}_i - \mathbf{c}_j)^2}$$

in the correlator (2, 6) cannot be expanded in the series of the fluctuations of immovable quasi-particles variables $(i, j > 2)$ with respect to the saddle-point, the integration over these fluctuations $(\delta\mathbf{c}_i, \delta\mathbf{c}'_i, i > 2)$ is Gaussian and can be correctly performed by the means of transition from the path integral in $\delta\mathbf{c}'_i$ $(i > 2)$ to the integration over $\delta\mathbf{c}'_{i||}, \delta\mathbf{c}'_{i\perp}$ fields that represent the longitudinal and transversal projections of $\delta\mathbf{c}'_i$ field on the $\delta\mathbf{c}_i$ direction.

The consistent analysis of higher variations of the action (17) around the saddle-point demands their study in the large T limit since this region contributes mainly to the integration over T. Using the explicit solution (32, 33) one can show that higher variations of the action are finite in the large T_β limit. This assures the consistency of the instanton approach and the existence of the fluctuation integral.

6. The integration over the scale parameter

Finally, the result of the instanton approach is

$$\ln Z^{(N)}_{\varphi^n} \sim \lim_{L\to 0} \frac{\partial}{\partial L} \operatorname*{residue}_{\varepsilon=0} \left\{ K^L \int_0^\infty \frac{\mathrm{d}y}{y} \left(\frac{D(y)}{2}\right)^N \kappa(y) \prod_{\beta=1}^{L} A^{[2]}_\beta \right\}. \tag{40}$$

The factor K^L arises from the L-dimensional integration over T. The function $\kappa(y)(\kappa(0) < \infty)$ results from the fluctuation integrations, besides the factor $\prod_\beta \mathrm{e}^{\mathrm{i}|\mathbf{k}|y}$ is also included in κ. The factor $1/y$ is extracted from the fluctuation integrations without explicit calculation by the means of the dimension analysis. Indeed, the value $\ln Z_{\varphi^n}$ is determined by logarithmic divergences of the diagrams of $\langle\varphi^2\varphi' \ldots \varphi'\rangle^{(N)}$. This gives the logarithmic behaviour in y of the integral (40). The convergence of (40) for large y is assured by $\kappa(y)$. Let us remind that this factor depends also on L.

Using formula (39), we note that $I_\triangle(\mathbf{x}_\beta) \sim T_\beta$ at the saddle-point, so (36) can be rewritten as

$$A^{[2]} = \sum_{j=0}^{\infty} \frac{1}{j!} \left(-\frac{K(\varepsilon)N(1 - y^\varepsilon f(\varepsilon))}{\varepsilon} \right)^j$$

with $K(\varepsilon)$ and $f(\varepsilon)$ being regular functions in ε, $K(0) \neq 0$, $f(0) = 1$. The extraction of UV divergences (i.e., poles in ε) has to be executed in the framework of the formalism developed in [14]. The parameter $N\varepsilon$ is supposed to be small. Using the explicit formula for $D(y)$ (34) the divergences at $\varepsilon \to 0$ can be easily extracted by integration by parts

$$\int_0^\infty \frac{\mathrm{d}y}{y} y^{N\varepsilon} \kappa(y) \frac{(1-y^\varepsilon)^j}{\varepsilon^j} = \frac{\kappa(0)}{\varepsilon^{j+1}} \sum_{s=0}^{j} \frac{(-1)^{j-s} j!}{(N+s)s!(j-s)!}.$$

Due to the choice of $\mathbf{D_v}$ correlator (see definition of D_0 constant in (7)) there are no $N\varepsilon$ corrections in (40). Then $A^{[2]}$ term does not affect the position and the type of the singularity and contributes in the large N limit to the amplitude of the asymptotes only. The final result for the simple pole in ε of the renormalization constant series in u reads

$$\ln Z_{\varphi^n}^{(N)} \sim \left\{ -\frac{\alpha}{(4\pi)^{d/2}\Gamma(d/2)d} \right\}^N. \tag{41}$$

7. Resummation of anomalous dimensions

We proceed now to the critical exponents Δ_{φ^n} analysis. The general expression for critical exponents is

$$\Delta_{\varphi^n} = d_{\varphi^n} + \gamma_{\varphi^n} = n(-1+\varepsilon) + \gamma_{\varphi^n},$$

where d_{φ^n} and γ_{φ^n} stand for the canonical and anomalous dimensions of the composite operator φ^n. Our main goal is to analyse the expansion of the anomalous dimension γ_{φ^n} in ε using the information about the series singularities. The latter are of the main interest since they can be used to resum the series. General theory [3] provides

$$\gamma_{\varphi^n} = D_\mu \ln Z_{\varphi^n} = \beta(u) \frac{\partial}{\partial u} \operatorname*{residue}_{\varepsilon=0} [\ln Z_{\varphi^n}] \bigg|_{u=u_*}$$
$$= -2u \frac{\partial}{\partial u} \operatorname*{residue}_{\varepsilon=0} [\ln Z_{\varphi^n}] \bigg|_{u=u_*}, \qquad u_* = 2\varepsilon \frac{(4\pi)^{d/2}\Gamma(d/2)d}{(d-1+\alpha)}, \tag{42}$$

where u_* is a fixed point that is known exactly for Kraichnan model.

The substitution of (41) into formula (42) gives the transformation of u-expansion of the anomalous dimension γ_{φ^n} to the series in ε considered at the large order of perturbation theory

$$\gamma_{\varphi^n} = \sum_{N\geqslant 0} \gamma_{\varphi_n}^{(N)} \varepsilon^N, \qquad \gamma_{\varphi^n}^{(N)} \sim \left[\frac{-2\alpha}{(d-1+\alpha)} \right]^N.$$

The absence of a power term N^b in the asymptote (i.e., $b = 0$ in (3)) indicates the simple pole singularity. The position of the nearest singularity ε_c in the anomalous dimension expansion in ε (which also gives the convergence radius R_c) reads

$$\varepsilon_c = -\frac{d-1+\alpha}{2\alpha}, \qquad R_c = |\varepsilon_c|.$$

This is used to resum the γ_{φ^n} series: we extract the singularity of the expansion adopting simple rational representation

$$\gamma_{\varphi^n} = \sum_{k=1}^{\infty} \gamma_{\varphi^n}^{(k)} \varepsilon^k = \frac{\sum_{k=1}^{\infty} \widetilde{\gamma}_{\varphi^n}^{(k)} \varepsilon^k}{\varepsilon - \varepsilon_c}.$$

The new coefficients $\widetilde{\gamma}_{\varphi^n}^{(k)}$ are calculated via the term by term comparison of the expansions giving the following coefficients ($n = 1, 2$)

$$\widetilde{\gamma}_{\varphi^n}^{(1)} = -\gamma_{\varphi^n}^{(1)} \varepsilon_c, \qquad \widetilde{\gamma}_{\varphi^n}^{(2)} = \gamma_{\varphi^n}^{(1)} - \gamma_{\varphi^n}^{(2)} \varepsilon_c.$$

We now need the values of the coefficients $\gamma_{\varphi^n}^{(k)}$ to carry out the resummation. The anomalous dimension γ_{φ^n}, $n > 1$ is calculated up to the second order in [8] and the coefficients are given by the formulae

$$\gamma_{\varphi^n}^{(1)} = -\frac{\alpha n(n-1)d}{d-1+\alpha},$$

$$\gamma_{\varphi^n}^{(2)} = 2\frac{\alpha(\alpha-1)n(n-1)(d-1)}{(d-1+\alpha)^2} + \frac{\alpha^2 n(n-1)(n-2)\,\mathrm{d}h(d)}{(d-1+\alpha)^2},$$

$$h(d) = \sum_{k=0}^{\infty} \frac{k!}{4^k(1+d/2)\ldots(k+d/2)} = F(1,1;1+d/2;1/4),$$

here F is the generalized hyper-geometric function. The resummation is then straightforward giving the following coefficients $\widetilde{\gamma}_{\varphi^n}^{(k)}$, $k = 1, 2$

$$\widetilde{\gamma}_{\varphi^n}^{(1)} = -n(n-1)\frac{d}{2}, \qquad \widetilde{\gamma}_{\varphi^n}^{(2)} = -n(n-1) + \frac{\alpha(n-2)\,\mathrm{d}h(d)}{2(d-1+\alpha)},$$

and the anomalous dimension

$$\gamma_{\varphi^n} = -\frac{2\alpha n(n-1)\varepsilon}{d-1+\alpha(1+2\varepsilon)}\left[\frac{d}{2} + \varepsilon - \frac{\alpha d(n-2)h(d)}{2(d-1+\alpha)}\varepsilon + O(\varepsilon^2)\right]. \tag{43}$$

Fortunately, this result can be verified: for $n = 2$ the exact value of the anomalous dimension is known

$$\gamma_{\varphi^2} = -\frac{2\varepsilon\alpha(d+2\varepsilon)}{d-1+\alpha(1+2\varepsilon)},$$

while the resummation (43) for anomalous dimension γ_{φ^2}, $n = 2$ yields

$$\gamma_{\varphi^2} = -\frac{2\alpha\varepsilon[d+2\varepsilon+O(\varepsilon^2)]}{d-1+\alpha(1+2\varepsilon)}.$$

We see that no high-order corrections to the resummation result occur in this case; resummation gives the exact answer, the singularity found for Z_{φ^2} is unique. This case illustrates that information about the type and the position of the singularity allows us appreciably supplement the information provided by the direct perturbation expansion.

8. Conclusions

This paper demonstrates that the instanton approach applies to the large-order analysis of the dynamic model. We can state that the study of the behaviour of the convergent series is a more difficult problem than that for the divergent ones. Using the instanton approach one can state that the perturbation series for the scaling dimensions of the operators considered are convergent with the finite convergence radius calculated. In each case the information about singularity type of the series permitted to accelerate considerably the convergence of the series.

Concerning the saturation problem [11] one can say that our results do not prove its existence. It is possible that two orders of the ε expansion are not sufficient to observe the saturation. Another possibility is that at large n the series for critical indices acquires an essential singularity. Indeed, we show the radius of convergence R_c is not depending on n. But in our framework, we cannot discuss the poles in ε outside the circle of convergence and theirs behaviour at $n \to \infty$.

Nevertheless, the value of indices obtained is of importance as an improved result of ε expansion.

Acknowledgments

The work was partially supported by Russian Foundation for Basic Research (grant no 05-02-17524) 2005-2008.

References

[1] Obukhov A M 1949 *Izv. Akad. Nauk SSSR, Ser. Geogr. Geofiz.* **13** 58
[2] Kraichnan R M 1968 *Phys. Fluids* **11** 945
[3] Zinn-Justin J 1989 *Quantum Field Theory and Critical Phenomena* (Oxford: Oxford University Press)
[4] Elperin T, Kleorin N and Rogachevskii I 1995 *Phys. Rev.* E **52** 2617
Elperin T, Kleorin N and Rogachevskii I 1997 *Phys. Rev.* E **55** 2713
Elperin T, Kleorin N and Rogachevskii I 1996 *Phys. Rev. Lett.* **76** 224
[5] Vergassola M and Avellaneda M 1997 *Physica* D **106** 148
[6] Chertkov M, Kolokolov I and Vergassola M 1997 *Phys. Rev.* E **56** 5483
Chertkov M, Kolokolov I and Vergassola M 1998 *Phys. Rev. Lett.* **80** 512
[7] Vergassola M and Mazzino A 1997 *Phys. Rev. Lett.* **79** 1849
[8] Adzhemyan L Ts and Antonov N V 1998 *Phys. Rev.* E **58** 7381
[9] Gawedzki K and Vergassola M 2000 *Physica* D **138** 63
[10] Balkovsky E, Falkovich G and Fouxon A 2001 *Phys. Rev. Lett.* **86** 2790
[11] Chertkov M 1997 *Phys. Rev.* E **55** 2722
[12] Balkovsky E and Lebedev V 1998 *Phys. Rev.* E **58** 5776
[13] Martin P C, Siggia E D and Rose H A 1973 *Phys. Rev.* A **8** 423
[14] Komarova M V and Nalimov M Yu 2001 *Theor. Math. Phys.* **126** 393
[15] Lipatov L N 1977 *Zh. Eksp. Teor. Fiz.* **72** 411
[16] Celani A, Lanotte A, Mazzino A and Vergassola M 2000 *Phys. Rev. Lett.* **84** 2385